Let us consider minimal propositional logic on some fixed set of propositional variables $P_1, P_2, \ldots, P_n$. The only typing rules we shall consider are the following ones :

$$\textbf{Prod} \quad \frac{\Gamma \vdash A : \texttt{Prop} \quad \Gamma \vdash B : \texttt{Prop}}{\Gamma \vdash A \to B : \texttt{Prop}}$$

$$\textbf{Lam} \quad \frac{\Gamma :: (v : A) \vdash e : B}{\Gamma \vdash [v\!:\!A]\,e : A \to B}$$

$$\textbf{App*} \quad \frac{E, \Gamma \vdash e : A_1 \to A_2 \to \ldots \to A_n \to B \quad E, \Gamma \vdash e_i : A_i \ \ (i = 1 \ldots n)}{E, \Gamma \vdash (e\, e_1\, e_2 \ldots e_n) : B}$$

$$\textbf{Var} \quad \frac{(x, A) \in E \cup \Gamma}{\Gamma \vdash x : A}$$

We want to prove that, if P is a proposition and π is a term of type P, then some combination of `intro`, `apply` and `assumption` can solve the goal P.

More generally, let us prove by induction on π that, if there is some context Γ and proposition A such that the judgment $\Gamma \vdash \pi : A$ is derivable from the rules **Lam**, **App*** and **Var**, then the tactics above can solve the goal $\Gamma \overset{?}{\vdash} A$ founding the solution π. In this proof, we consider only proof terms in β-normal form. Then all applications $(x\ e_1\ \ldots e_n)$ are such that x is a variable. This is not a restriction, since we can convert any term of some type P into such a form having the same type (look for instance at *Coq* reference manual or any book on simply typed lambda-calculus).

Our proof deals with three cases :

- If π is a variable, then π is declared of type A in Γ ; this solution is solved by a call to `assumption`.
- If π is an abstraction $[v\!:\!B]\pi'$, then we have the judgment $\Gamma :: (v : B) \vdash \pi' : C$, where A is $B \to C$. By induction hypothesis, the goal $\Gamma :: (v : B) \overset{?}{\vdash} C$ can be solved (with solution π') using a combination τ of calls to our elementary tactics. Then the tactic "`intro` v `;`τ" solves our goal, producing the solution π.
- If π is an application $(x\ \pi_1\ \pi_2\ \ldots \pi_n)$, where $\Gamma \vdash \pi_i : A_i$ and $\Gamma \vdash x : A_1 \to A_2 \to \ldots A_n \to B$.
  By induction hypothesis, each subgoal $\Gamma \overset{?}{\vdash} A_i$ can be solved (with respective solutions π_i) using combinations τ_i of calls to our elementary tactics. Then the tactic "`apply` x `;[`τ_1`|`$\tau_2 \ldots$`|`τ_n`]`" solves our goal with π as the solution.

Notes

This result implies that, if some goal is solvable in minimal propositional logic, then "`auto` n" can solve it, for a suitable value of n.

(More) exercises

- Give a solution of this exercise without using composed tactics.
- What are normal forms used for ? Which result could we prove without using them ?